Graph Rewriting Formal Definitions

The Design and Implementation of a Graph Rewrite Engine for Model Transformations. Kimmo Nupponen. 2005.

1. A *graph* is $\langle \Sigma, V, E, \text{vlab}, \text{elab} \rangle$
   a. $\Sigma = \Sigma_V \cup \Sigma_E$ is a set of labels.
   b. V is the set of vertices.
   c. $E : V \times V$ is the set of edges.
   d. $\text{vlab} : V \longrightarrow \Sigma_V$ is the vertex-labeling function.
   e. $\text{elab} : V \longrightarrow \Sigma_E$ is the edge-labeling function.

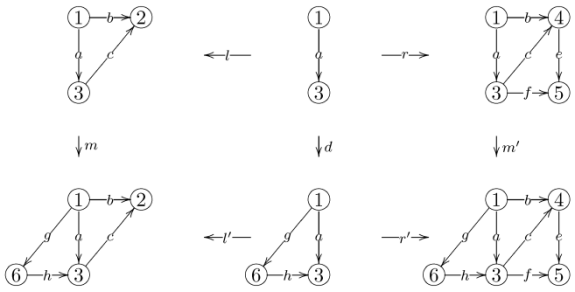

Figure 1. Rule application example

2. Auxiliary notations
   a. $\mathcal{L}^{\Sigma}$ is the set of all graphs over Σ.
   b. $\text{source} : E \longrightarrow V$. $\text{source}(e)$ returns the source node of $e \in E$.
   c. $\text{target} : E \longrightarrow V$. $\text{target}(e)$ returns the target node of $e \in E$.

3. A *(total) graph morphism* m from $G \in \mathcal{L}^{\Sigma}$ to $H \in \mathcal{L}^{\Sigma}$ is denoted by $m : G \to H$
   a. m is a pair of functions. $m = (m_V : V_G \longrightarrow V_H, m_E : E_G \longrightarrow E_H)$
   b. m preserves structure and labels.
      i. $\text{source}_H \circ m_E = m_V \circ \text{source}_G$
      ii. $\text{target}_H \circ m_E = m_V \circ \text{target}_G$
      iii. $\text{vlab}_H \circ m_V = \text{vlab}_G$
      iv. $\text{elab}_H \circ m_E = \text{elab}_G$

4. A *rewrite rule* is $\langle L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R \rangle$
   a. $L \in \mathcal{L}^{\Sigma}$ is the *left hand side*; $R \in \mathcal{L}^{\Sigma}$ is the *right hand side*.
   b. $K \in \mathcal{L}^{\Sigma}$ is the "glue graph" for embedding.
   c. $l : K \longrightarrow L$
   d. $r : K \longrightarrow R$

5. Let $p = \langle L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R \rangle$ be a rule. Let $G \in \mathcal{L}^{\Sigma}$. The total graph morphism $m : L \to G$ is called a *match of* L *in* G. m also defines the occurrence of p in G.

6. $H \in \mathcal{L}^{\Sigma}$ is *directly derivable* from $G \in \mathcal{L}^{\Sigma}$ using rule $p = \langle L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R \rangle$, written as $G \overset{p}{\Rightarrow} H$, if and only if there exist D, m, d, m', l', r' such that: (arrows represent total graph morphism)

$$\begin{array}{ccccc} L & \xleftarrow{\ l\ } & K & \xrightarrow{\ r\ } & R \\ {\scriptstyle m}\downarrow & & {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle m'} \\ G & \xleftarrow{\ l'\ } & D & \xrightarrow{\ r'\ } & H \end{array}$$

7. A *graph rewriting system* is a set of rules.